JOURNAL OF ALGEBRA 12, 494–524 (1969)

A Characterization of the Finite Simple Groups $PSp_6(q)$, q Odd*

W. J. WONG

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556

Communicated by Walter Feit

Received September 5, 1968

If V is a vector space with a nondegenerate alternating bilinear form (more briefly, a symplectic vector space) of dimension 6 over a field F of characteristic different from 2, we can express V as an orthogonal direct sum of a 2-dimensional subspace W and a 4-dimensional subspace X, and consider the involution T of the symplectic group $Sp(V) = Sp_6(F)$ which fixes the elements of X and multiplies the elements of W by -1. Then the centralizer of T in $Sp_6(F)$ consists of all elements which leave both W and X invariant, and is the direct product of the subgroup L of elements fixing all the vectors of X with the subgroup J of elements fixing all the vectors of W. By their actions on W and X, we see that L and J are respectively isomorphic with $Sp_2(F) = SL_2(F)$ and $Sp_4(F)$. If t is the involution of the projective symplectic group $PSp_6(F)$ corresponding to T, then since T is not conjugate in $Sp_6(F)$ to $-T$, the centralizer of t in $PSp_6(F)$ is the subgroup corresponding to the centralizer of T in $Sp_6(F)$ under the natural homomorphism of $Sp_6(F)$ on $PSp_6(F)$. Since they do not contain $-I$, L and J are mapped isomorphically by this homomorphism onto subgroups $\bar{L}$, $\bar{J}$ of $PSp_6(F)$, whose product is the centralizer of t, such that $[\bar{L}, \bar{J}] = 1$. The product is not direct, since the centers of L and J, which are generated by T and $-T$ respectively, are mapped by the homomorphism on the same subgroup of $PSp_6(F)$. Thus the centralizer of t in $PSp_6(F)$ is the central product of two groups isomorphic with $SL_2(F)$ and $Sp_4(F)$ respectively.

In this paper we prove that when F is finite then this property of $PSp_6(F)$ characterizes it among all finite simple groups. More precisely, we prove the following result.

* This research was partially supported by the National Science Foundation grant GP-8366 at the University of Notre Dame.

THEOREM. *Let G be a finite group with an involution t whose centralizer in G is given by*

$$C(t) = LJ, \qquad [L, J] = 1, \qquad L \cap J = \langle t \rangle,$$

where $L \approx SL_2(q)$, $J \approx Sp_4(q)$, *for some odd* q. *Then either*

(i) $G = C(t)\, O(G)$, *or*

(ii) *G is isomorphic with* $PSp_6(q)$.

Here $O(G)$ denotes the largest normal subgroup of odd order in G. In particular, $PSp_6(q)$ is the only simple group satisfying the hypothesis of the theorem.

Our method is similar to that used in our earlier characterization of $PSp_4(q)$ [*12*]. In §1 we study the 2-structure of G, i.e. the involutions and 2-subgroups of G, beginning from knowledge of the 2-structure of $C(t)$. In particular, if case (i) of the conclusion of the theorem does not hold, then we prove that G has exactly two classes of involutions. We also determine the structure of the normalizer in G of a four-subgroup D whose involutions are all conjugate to t. Because of the existence of this subgroup D, we do not have to determine the structure of the centralizer in G of an involution which is not conjugate to t, in contrast to the situation in the characterization of $PSp_4(q)$. However, we do find the order of this centralizer, by using a result of Brauer.

In §2, we study the p-structure of G, where p is the prime divisor of q, beginning from knowledge of the p-structure of $C(t)$. In particular, we find the structure of the normalizer in G of a Sylow p-subgroup. We are then able, in §3, to construct a (BN)-pair [*11*], and so to construct a certain subgroup G_0 of G whose multiplication table is uniquely determined. By using a lemma of Suzuki, we prove that $G_0 = G$, from which it follows that G is isomorphic with $PSp_6(q)$.

Our notation is largely standard. We use $O(X)$ to denote the largest normal subgroup of odd order in the finite group X. The normalizer and centralizer of Y in X are denoted $N_X(Y)$ and $C_X(Y)$; we omit the subscript when $X = G$. We write $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$. If $x^y = z$, we also write $y : x \rightarrow z$; $y : x \leftrightarrow z$ means that $y : x \rightarrow z$ and $y : z \rightarrow x$. If $y : x \rightarrow x^{-1}$, we say that y inverts x. The field of q elements is denoted F_q. We shall take linear transformations on a vector space as acting on the right, so that if θ is a linear transformation, its matrix with respect to a basis $\{e_1, e_2, \ldots\}$ is (a_{ij}), where $e_i\theta = \sum_j a_{ij}e_j$. A symplectic basis of a 4-dimensional symplectic vector space is a basis $\{e_1, e_2, e_3, e_4\}$ such that $(e_1, e_2) = (e_3, e_4) = 1$, $(e_i, e_j) = 0$ if $i = 1, 2$ and $j = 3, 4$. Finally, I will always denote the identity matrix of degree 2.

1. The 2-Structure of G

We shall assume throughout the paper that G is a finite group satisfying the hypothesis of the theorem:

G has an involution t_1 such that

$$C(t_1) = L_1 J, \qquad [L_1, J] = 1, \qquad L_1 \cap J = \langle t_1 \rangle, \tag{1}$$

where $L_1 \approx SL_2(q)$, $J \approx Sp_4(q)$, and q is odd.
Here we have placed a subscript on t and L, for reasons which will soon appear. We shall assume also that

$$G \neq C(t_1)\, O(G). \tag{2}$$

Our aim is to prove that $G \approx PSp_6(q)$.

We shall identify J with $Sp_4(q)$, and in turn identify the elements of $Sp_4(q)$ with their matrices with respect to a symplectic basis $\{e_1, e_2, e_3, e_4\}$ of the 4-dimensional symplectic vector space over F_q on which they act. Thus J consists of all 4×4 matrices M over F_q such that $MBM' = B$, where M' is the transpose of M and

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since $t_1 \in Z(J)$, the matrix representation of t_1 as an element of J is

$$t_1 = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}.$$

We now choose two other involutions of J,

$$t_2 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \qquad t_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

so that

$$t_1 t_2 t_3 = 1 \tag{3}$$

and we have a four-subgroup

$$D = \{1, t_1, t_2, t_3\} = \langle t_1, t_2 \rangle.$$

We identify $SL_2(q)$ with the group of all 2×2 matrices over F_q with determinant 1, and set

$$L_2 = \left\{ \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \middle| \, A \in SL_2(q) \right\}, \qquad L_3 = \left\{ \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \middle| \, A \in SL_2(q) \right\}.$$

Then L_2 and L_3 are subgroups of J, and $C_J(t_2)$ is their direct product, so that

$$C(D) = L_1L_2L_3\,. \tag{4}$$

Also, we have isomorphisms

$$\varphi_2 : SL_2(q) \to L_2\,,\ \varphi_2(A) = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix},$$

$$\varphi_3 : SL_2(q) \to L_3\,,\ \varphi_3(A) = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}.$$

We also choose an isomorphism

$$\varphi_1 : SL_2(q) \to L_1\,.$$

Thus, for $i = 1, 2, 3$, we have

$$t_i = \varphi_i(-I).$$

We define integers δ, n, e by the conditions

$$q - \delta = 2^n e, \qquad n \geqslant 2, \qquad \delta = \pm 1, \qquad e \text{ odd},$$

so that, in particular, $q \equiv \delta(\text{mod } 4)$. The order of $C(t_1)$ is

$$|\,C(t_1)| = \tfrac{1}{2}\,|\,L\,|\,|\,J\,| = \tfrac{1}{2}q(q^2-1)q^4(q^4-1)\,(q^2-1),$$

so that a Sylow 2-subgroup of $C(t_1)$ has order 2^{3n+3}. We now proceed to construct such a subgroup. We fix a generator ϵ of the multiplicative group of F_q, and, for $i = 1, 2, 3$, set

$$d_i = \varphi_i \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} (\delta = 1), \qquad \text{or} \qquad d_i = \varphi_i \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} (\delta = -1),$$

where in the second case α and β are elements of F_q such that $\alpha + \beta\sqrt{-1}$ is a generator of the group of elements of F_{q^2} whose norm in F_q is 1. Then d_i is an element of order $q - \delta$ in L_i, generating a subgroup whose normalizer is $\langle d_i\,, b_i \rangle$, where

$$b_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\delta = 1), \qquad \text{or} \qquad b_i = \varphi_i \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix} (\delta = -1),$$

where in the second case λ and μ are elements of F_q such that $\lambda^2 + \mu^2 = -1$. Then b_i inverts d_i, and $b_i^2 = t_i$. In fact,

$$N_{L_i}(\langle d_i^m \rangle) = \langle d_i\,, b_i \rangle, \qquad \text{if} \qquad d_i^m \neq 1, t_i\,. \tag{5}$$

If we put

$$a_i = d_i^e,$$

then $Q_i = \langle a_i\,, b_i \rangle$ is a Sylow 2-subgroup of L_i, and is a generalized quaternion group of order 2^{n+1}. Because of (3),

$$|\,Q_1Q_2Q_3\,| = \tfrac{1}{2}\,|\,Q_1\,|\,|\,Q_2\,|\,|\,Q_3\,| = 2^{3n+2}.$$

To obtain a Sylow 2-subgroup of $C(t_1)$ we adjoin the element

$$u = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

of J, which has the properties

$$u^2 = t_1, \qquad u : \varphi_1(x) \to \varphi_1(x), \qquad \varphi_2(x) \leftrightarrow \varphi_3(x), \tag{6}$$

for all x in $SL_2(q)$. Then we see that a Sylow 2-subgroup S of $C(t_1)$ is given by

$$S = \langle Q_1, Q_2, Q_3, u \rangle,$$

$$Q_i = \langle a_i, b_i \rangle, a_i^{2^{n-1}} = b_i^2 = (a_i b_i)^2 = t_i, t_i^2 = 1 \qquad (i = 1, 2, 3),$$

$$[Q_1, Q_2] = [Q_1, Q_3] = [Q_2, Q_3] = 1, t_1 t_2 t_3 = 1,$$

$$u^2 = t_1, [Q_1, u] = 1, a_2^u = a_3, b_2^u = b_3.$$

Since $Z(Q_i) = \langle t_i \rangle$, and u interchanges t_2 and t_3, we see that

$$Z(S) = \langle t_1 \rangle. \tag{7}$$

Since t_1 is the only involution of L_1, and J has only two classes of involutions, represented by t_1 and t_2 ([4], p. 102), every involution of $C(t_1)$ is conjugate in $C(t_1)$ to t_1 or t_2, or is of the form xy, where $x \in L_1$, $y \in J$, and $x^2 = y^2 = t_1$. Since L_1 and J each has only one class of elements whose square is t_1 ([4], p. 106), all these involutions are conjugate in $C(t_1)$. One such involution is

$$v = f_1 f_2 f_3,$$

where, for $i = 1, 2, 3$,

$$f_i = a_i^{2^{n-2}} = \varphi_i \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} (\delta = 1), \qquad \text{or} \qquad \varphi_i \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} (\delta = -1),$$

where in the first case $\lambda^2 = -1$ and in the second case $\lambda = \pm 1$. Thus, $C(t_1)$ has 3 classes of involutions, represented by t_1, t_2 and v.

We remark that $C_{L_i}(f_i) = \langle d_i \rangle$, and that b_i transforms f_i into $t_i f_i$. Hence

$$C_{L_1}(v) = \langle d_1 \rangle, \quad v^{b_1} = t_1 v, \quad v^{b_2 b_3} = t_2 t_3 v = t_1 v.$$

Since the centralizer in $C(t_1)$ of any element z consists of all elements xy, where $x \in L_1$, $y \in J$, and either $z^x = z^y = z$, or $z^x = z^y = t_1 z$, we have

$$C(t_1, v) = \langle d_1, C_J(v), b_1 b_2 b_3 \rangle. \tag{8}$$

An easy computation shows that $C_J(v) = C_J(f_2f_3)$ consists of all elements of the form

$$\begin{pmatrix}\alpha & 0 & \beta & 0\\ 0 & \alpha' & 0 & \beta'\\ \gamma & 0 & \mu & 0\\ 0 & \gamma' & 0 & \mu'\end{pmatrix}(\delta = 1), \qquad \text{or} \qquad \begin{pmatrix}\alpha & \beta & \gamma & \mu\\ -\beta & \alpha & -\mu & \gamma\\ \rho & \sigma & \tau & \lambda\\ -\sigma & \rho & -\lambda & \tau\end{pmatrix}(\delta = -1),$$

where in the first case $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \mu\end{smallmatrix}\right)$ lies in $GL_2(q)$ and $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \mu'\end{smallmatrix}\right)$ is its contragredient (inverse transpose), and in the second case

$$\begin{pmatrix}\alpha + \beta\sqrt{-1} & \gamma + \mu\sqrt{-1}\\ \rho + \sigma\sqrt{-1} & \tau + \lambda\sqrt{-1}\end{pmatrix}$$

is an element of $GU_2(q)$, the group of all unitary matrices of degree 2 with coefficients in F_{q^2}. Hence,

$$C_J(v) \approx GL_2(q)\ (\delta = 1), \text{ or } GU_2(q)\ (\delta = 1),$$

and so

$$|C_J(v)| = q(q^2 - 1)(q - \delta).$$

In either case, $\langle a_2, a_3, u\rangle$ is a Sylow 2-subgroup of $C_J(v)$.

The derived group $C_J(v)'$ is the subgroup of $C_J(v)$ corresponding to $SL_2(q)$ or $SU_2(q)$, and has order $q(q^2 - 1)$. It contains the element $[a_3, u] = a_2a_3^{-1}$, and also contains u, since u is the element of $C_J(v)$ corresponding to the element $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ of $GL_2(q)$ or $GU_2(q)$. Thus the generalized quaternion group $\langle a_2a_3^{-1}, u\rangle$ of order 2^{n+1} is a Sylow 2-subgroup of $C_J(v)'$. Since d_2 is an element of $C_J(v)$ corresponding to an element of $GL_2(q)$ or $GU_2(q)$ whose determinant is a primitive $(q - \delta)$-th root of 1, $C_J(v)$ is the semi-direct product of $\langle d_2\rangle$ and $C_J(v)'$.

LEMMA 1.1. *If X is a subgroup of G, T is a Sylow 2-subgroup of $X \cap C(t_1)$, and $\langle t_1\rangle$ is a characteristic subgroup of T, then T is a Sylow 2-subgroup of X. In particular, S is a Sylow 2-subgroup of G.*

Proof. Let U be a Sylow 2-subgroup of X containing T. Then $N_U(T)$ normalizes the characteristic subgroup $\langle t_1\rangle$ of T, so that $N_U(T) \leqslant C(t_1)$. Thus $N_U(T) = T$, so that $U = T$, by an elementary property of groups of prime power order. This proves the first statement. The second statement follows by setting $X = G$, $T = S$, since $\langle t_1\rangle$ is characteristic in S, by (7).

LEMMA 1.2. *G has exactly two conjugacy classes of involutions, K_1 and K_2, where $K_1 \cap C(t_1)$ consists of the classes in $C(t_1)$ represented by t_1 and t_2, and $K_2 \cap C(t_1)$ is the class in $C(t_1)$ represented by v.*

Proof. From (8), a Sylow 2-subgroup of $C(t_1, v)$ is the group

$$T = \langle a_1, a_2, a_3, u, b_1b_2b_3\rangle,$$

or order 2^{3n+1}. We have the commutator relations

$$[a_1, a_2] = [a_1, a_3] = [a_2, a_3] = [a_1, u] = [u, b_1b_2b_3] = 1,$$

$$[u, a_2] = [a_3, u] = a_2a_3^{-1}, [b_1b_2b_3, a_i] = a_i^2 \qquad (i = 1, 2, 3).$$

It follows that

$$T' = \langle a_1^2, a_2^2, a_3^2, a_2a_3^{-1}\rangle,$$

an Abelian group. The set of all 2^{n-1}-th powers of elements of T' is the subgroup $\langle t_1\rangle$, so that $\langle t_1\rangle$ is characteristic in T. By Lemma 1.1, T is a Sylow 2-subgroup of $C(v)$. Since T is not isomorphic with the Sylow 2-subgroup S of $C(t_1)$, v is not conjugate to t_1 in G.

By a theorem of Glauberman ([*6*], Th. 1), the condition (2) is equivalent with saying that t_1 is conjugate in G to some involution of S distinct from t_1. Since such an involution is conjugate in $C(t_1)$ to v or t_2, t_1 must be conjugate to t_2 in G. This proves the lemma.

The next result is fundamental for the rest of the paper.

LEMMA 1.3. *Let* $D = \langle t_1, t_2\rangle$. *Then*

$$N(D) = \langle L_1L_2L_3, s, u\rangle,$$

where $s^2 = t_3$, $(su)^3 = 1, L_1^s = L_2$, *and* $[s, L_3] = 1$.

Proof. By Lemma 1.2, all involutions of $C(t_1)$ distinct from t_1 which lie in K_1 are conjugate in $C(t_1)$. Since t_2 is conjugate in G to t_1, all involutions of $C(t_2)$ distinct from t_2 which lie in K_1 are conjugate in $C(t_2)$. Thus there exists an element y such that

$$y \in C(t_2), \qquad y : t_1 \to t_3 .$$

Then $y : t_3 = t_1t_2 \to t_3t_2 = t_1$, so that $y \in N(D)$. We set

$$s = u^y.$$

Since $u^2 = t_1$, $s^2 = t_1^y = t_3$. Since u interchanges t_2 and t_3, s interchanges $t_2^y = t_2$ and $t_3^y = t_1$. We see that $N(D)$ induces the full automorphism group on D, so that

$$N(D)/C(D) \approx \Sigma_3,$$

the symmetric group of degree 3. Also,

$$N(D) = \langle C(D), s, u\rangle = \langle L_1L_2L_3, s, u\rangle,$$

by (4). Now, y normalizes $C(D) = L_1L_2L_3$ and so acts on the quotient group

$$C(D)/D = L_1D/D \times L_2D/D \times L_3D/D.$$

The group $L_iD/D \approx L_i/L_i \cap D = L_i/\langle t_i\rangle \approx PSL_2(q)$ is an indecomposable group with trivial center. By the Krull–Schmidt Theorem, y permutes the subgroups L_1D, L_2D, L_3D among themselves. Since L_i is the unique subgroup of index 2 in L_iD, y permutes the subgroups L_1, L_2, L_3 among themselves. Since t_1, t_2, t_3 are the unique involutions in L_1, L_2, L_3 respectively, we must have

$$L_1{}^y = L_3, \qquad L_2{}^y = L_2, \qquad L_3{}^y = L_1.$$

Now, $L_1{}^s = (L_3{}^u)^y = L_2{}^y = L_2$, $[s, L_3] = [u, L_1]^y = 1$.

Suppose that $x \in L_2$. Since $x^s \in L_1$, $x^u \in L_3$, and s^2 and u^2 centralize $L_1L_2L_3$,

$$x^{sususu} = x^{ssusu} = x^{usu} = x^{uu} = x.$$

Thus, $[L_2, (su)^3] = 1$. Since $L_1 = L_2^{su}, L_3 = L_1^{su}$, and su centralizes $(su)^3$, we also have $[L_1, (su)^3] = [L_3, (su)^3] = 1$, so that

$$(su)^3 \in C(C(D)) = Z(C(D)) = D.$$

Since su permutes the involutions t_1, t_2, t_3 of D cyclically, we must have $(su)^3 = 1$. This proves the lemma.

We may assume that the isomorphism φ_1 of $SL_2(q)$ with L_1 is so chosen that $\varphi_1(x)^s = \varphi_2(x)$, for all x in $SL_2(q)$. Thus the properties of s analogous to (6) are

$$s^2 = t_3, \qquad s : \varphi_1(x) \leftrightarrow \varphi_2(x), \qquad \varphi_3(x) \rightarrow \varphi_3(x), \tag{9}$$

for all x in $SL_2(q)$, so that

$$su : \varphi_1(x) \rightarrow \varphi_3(x) \rightarrow \varphi_2(x) \rightarrow \varphi_1(x), \text{ all } x \text{ in } SL_2(q).$$

Now,

$$(usu)^{-1} = u^{-1}s^{-1}u^{-1} = ut_1t_3st_1u = ut_1t_3t_2su = usu,$$

by (3). Since $sususu = 1$, we see that

$$sus = usu. \tag{10}$$

We shall find the order of the centralizers of involutions in the class K_2. This information will be required only at the end of the proof of the theorem.

LEMMA 1.4. *$C(v)$ has a subgroup K of index 2, with Sylow 2-subgroup*

$$Y = \langle a_1 , a_2 , a_3 , u \rangle,$$

and $C_K(t_1) = \langle d_1 , C_J(v) \rangle$.

Proof. By the proof of Lemma 1.2, $C(v)$ has as Sylow 2-subgroup the group

$$T = \langle a_1 , a_2 , a_3 , u, b_1b_2b_3 \rangle.$$

Let T^* be the focal group of T in $C(v)$, i.e.

$$T^* = T \cap C(v)'.$$

Clearly T^* contains

$$T' = \langle a_1^2, a_2^2, a_2a_3 \rangle.$$

We have previously seen that $u \in C_J(v)'$, so that T^* contains the subgroup

$$W = \langle a_1^2, a_2^2, a_2a_3 , u \rangle.$$

The quotient group T/W is elementary Abelian of order 8,

$$T/W = \langle a_1W, a_2W, b_1b_2b_3W \rangle.$$

The group

$$Y = \langle a_1 , a_2 , a_3 , u \rangle$$

is a maximal subgroup of T, containing W. The involutions of Y have the form xy, where $x \in \langle a_1 \rangle$, $y \in \langle a_2 , a_3 , u \rangle$, and either $x^2 = y^2 = 1$ or $x^2 = y^2 = t_1$. If $x^2 = y^2 = 1$, then $xy \in K_1$, and if $x^2 = y^2 = t_1$, then $xy \in K_2$. Since f_1 commutes with x and f_2f_3 with y, $(xf_1)^2 = x^2f_1^2 = x^2t_1$ and $(yf_2f_3)^2 = y^2f_2^2f_3^2 = y^2t_2t_3 = y^2t_1$. Thus, if z is an involution of Y lying in K_2, then $zv = 1$ or $zv \in K_1$. The involutions $b_1b_2b_3$, $b_1b_2b_3a_1$, $b_1b_2b_3a_2$, $b_1b_2b_3a_1a_2$ lie in K_2. If one of them were conjugate in $C(v)$ to an involution z of Y, then $b_1b_2b_3v$, $b_1b_2b_3a_1v$, $b_1b_2b_3a_2v$ or $b_1b_2b_3a_1a_2v$ would be conjugate to zv, an element of K_1. This is impossible, since $b_1b_2b_3v$, $b_1b_2b_3a_1v$, $b_1b_2b_3a_2v$ and $b_1b_2b_3a_1a_2v$ lie in K_2.

We now apply a lemma of Thompson ([*10*], Lemma 5.38), which may be stated as follows:

LEMMA. *If Y is a maximal subgroup of a Sylow 2-subgroup of a finite group H, then every involution of H' is conjugate in H to an element of Y.*

In our case it follows that T^* does not contain $b_1b_2b_3$, $b_1b_2b_3a_1$, $b_1b_2b_3a_2$ or $b_1b_2b_3a_1a_2$. Since these four elements represent the cosets of W in T which do not lie in Y, it follows that $T^* \leq Y$, and hence that $C(v)$ has a subgroup K of index 2 with Y as Sylow 2-subgroup.

Since $\langle d_1, C_J(v)\rangle$ is the unique subgroup of index 2 in

$$C(t_1, v) = \langle d_1, C_J(v), b_1b_2b_3\rangle$$

having Y as Sylow 2-subgroup, we must have

$$C_K(t_1) = \langle d_1, C_J(v)\rangle.$$

This proves the lemma.

LEMMA 1.5. *K has a normal subgroup R of index 2^{n-1}, with*

$$X = \langle a_1a_2^{-1}, a_2a_3^{-1}, u\rangle$$

as Sylow 2-subgroup, and

$$C_R(t_1) = Z \times M,$$

where Z is cyclic of order $e = (q - \delta)/2^n$, and M is isomorphic with $GL_2(q)$ or $GU_2(q)$ according as $\delta = 1$ or $\delta = -1$. Also, R has no subgroup of index 2.

Proof. The focal group Y^* of Y in K contains

$$Y' = \langle a_2a_3^{-1}\rangle.$$

Since $u \in C_J(v)' \leqslant K'$, Y^* contains u also. Since $(a_2a_3^{-1})^{2^{n-1}} = t_2t_3 = t_1$, $\langle t_1\rangle$ is characteristic in Y, so that $N_K(Y) \leqslant C_K(t_1) = \langle d_1, C_J(v)\rangle$. Hence,

$$N_K(Y)' \cap Y \leqslant C_J(v)' \cap Y = \langle a_2a_3^{-1}, u\rangle.$$

By Grün's theorem, Y^* is generated by $\langle a_2a_3^{-1}, u\rangle$ and the elements of Y which are conjugate in K to elements of $\langle a_2a_3^{-1}\rangle$.

The subgroup

$$A = \langle a_1, a_2, a_3\rangle$$

is an Abelian maximal subgroup of Y. If B were another Abelian maximal subgroup of Y, then $Z(Y)$ would contain the subgroup $A \cap B$ of index 2 in A. However, $Z(Y) = \langle a_1, a_2a_3\rangle$ has index 2^n in A. Thus A is the only Abelian maximal subgroup of Y. If z_1, z_2 are elements of A which are conjugate in K, say

$$z_1 = z_2^x, \qquad x \in K,$$

then A and A^x are subgroups of $C_K(z_1)$. If T_1 and T_2 are Sylow 2-subgroups of $C_K(z_1)$ containing A and A^x respectively, then by Sylow's theorem there is an element y of $C_K(z_1)$ such that $T_1 = T_2^y$. Since A^{xy} is contained in T_1 but A is the only Abelian subgroup of order $|A|$ in T_1, we have $A^{xy} = A$, i.e. $xy \in N_K(A)$. Also, $z_1 = z_1^y = z_2^{xy}$. Thus two elements of A which are conjugate in K are already conjugate in $N_K(A)$.

The involutions of A lying in the class K_1 are t_1, t_2, t_3 and generate the four-group D. Thus $N(A) \leqslant N(D)$. By Lemma 1.3 and (5),

$$N(A) = \langle d_1, d_2, d_3, b_1, b_2, b_3, s, u\rangle,$$

so that

$$N(A) \cap C(v) = \langle d_1, d_2, d_3, b_1b_2b_3, s, u\rangle.$$

Since $N(A) \cap C(v) = N_K(A)\langle b_1b_2b_3\rangle$ and $b_1b_2b_3$ inverts $a_2a_3^{-1}$, an element of A is conjugate in $N_K(A)$ to an element of $\langle a_2a_3^{-1}\rangle$ if and only if it is conjugate in $N(A) \cap C(v)$ to an element of $\langle a_2a_3^{-1}\rangle$. Since the conjugacy class of $a_2a_3^{-1}$ in $N(A) \cap C(v)$ consists of all $a_ia_j^{-1}$ where $i, j = 1, 2, 3$, and $i \neq j$, the elements of A conjugate in K to an element of $\langle a_2a_3^{-1}\rangle$ generate the subgroup $\langle a_1a_2^{-1}, a_2a_3^{-1}\rangle$. Thus Y^* is generated by the subgroup

$$X = \langle a_1a_2^{-1}, a_2a_3^{-1}, u\rangle$$

of index 2^{n-1} in Y, together with the elements of $Y - A$ which are conjugate in K to an element of $\langle a_2a_3^{-1}\rangle$.

Let z be such an element of $Y - A$. We have

$$z = a_1^{\,i}a_2^{\,j}a_3^{\,k}u,$$

for some i, j, k, so that $z^2 = a_1^{2i}(a_2a_3)^{j+k}t_1$. If z is not an involution, then t_1 is the unique involution in both $\langle a_2a_3^{-1}\rangle$ and $\langle z\rangle$, so that z must be conjugate in $C_K(t_1)$ to an element of $\langle a_2a_3^{-1}\rangle$. Since $a_2a_3^{-1} \in C_J(v)' = C_K(t_1)'$, we see that

$$z \in C_J(v)' \cap Y = \langle a_2a_3^{-1}, u\rangle \leqslant X.$$

If z is an involution, then z must be conjugate in K to the involution t_1 of $\langle a_2a_3^{-1}\rangle$, so that z belongs to the class K_1. This implies that $a_1^{2i} = (a_2a_3)^{j+k}t_1 = 1$, so that $i \equiv 0 (\mathrm{mod}\ 2^{n-1})$ and $j + k \equiv 2^{n-1} (\mathrm{mod}\ 2^n)$. Thus, $a_3^{i+j+k} = 1$ or t_3. Since $t_3 = (a_1a_2^{-1})^{2^{n-1}} \in X$, and

$$z = (a_1a_2^{-1})^i\,(a_2a_3^{-1})^{i+j}\,a_3^{i+j+k}u,$$

we see that $z \in X$. Thus, $Y^* = X$, and K has a normal subgroup R of index 2^{n-1}, with X as Sylow 2-subgroup.

Since $C_J(v)$ is the semi-direct product

$$C_J(v) = \langle d_2\rangle\, C_J(v)',$$

$\langle d_1\rangle \approx \langle d_2\rangle$, and d_1 centralizes $C_J(v)'$, we see that

$$M = \langle d_1d_2^{-1}\rangle\, C_J(v)'$$

is a subgroup isomorphic to $C_J(v)$, i.e. to $GL_2(q)$ or $GU_2(q)$ according as $\delta = 1$ or $\delta = -1$. If Z is the subgroup of order e in $\langle d_1 \rangle$, then $Z \times M$ is the unique normal subgroup of index 2^{n-1} in $C_K(t_1)$ having X as Sylow 2-subgroup, and so we must have

$$C_R(t_1) = Z \times M.$$

Since R contains all elements of $C(v)$ of odd order, su lies in R. We know that u and $a_2a_3^{-1}$ lie in $C_J(v)' = M'$ and so in R'. Hence $a_1a_2^{-1} = (a_2a_3^{-1})^{su}$ also lies in R'. Thus $X \leqslant R'$, and so R has no subgroup of index 2. This proves the lemma.

LEMMA 1.6. $|C(v)| = q^3(q^3 - \delta)(q^2 - 1)(q - \delta)$.

Proof. If we set $\sigma_1 = a_1a_2^{-1}$, $\sigma_2 = a_1a_3^{-1}$, $\tau = ut_2$, then we see that the Sylow 2-subgroup X of R is generated by σ_1, σ_2 and τ, and that

$$\sigma_1^{2^n} = \sigma_2^{2^n} = \tau^2 = 1, \qquad [\sigma_1, \sigma_2] = 1, \qquad \sigma_1{}^\tau = \sigma_2,$$

so that X is the wreath product of a cyclic group of order 2^n with a group of order 2. We set $\beta = \sigma_1\sigma_2^{-1}$. We can choose a matrix representation of M as $GL_2(q)$ or $GU_2(q)$ so that τ is represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and β by $\left(\begin{smallmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{smallmatrix}\right)$, where γ is a primitive 2^nth root of 1, since $\beta = [\tau, \sigma_1]$ is an element of order 2^n lying in M', which is the subgroup of M corresponding to $SL_2(q)$ or $SU_2(q)$, and since τ is an involution inverting β. Now we can easily compute that $|C_M(\beta)| = |C_M(\tau)| = (q - \delta)^2$, $|C_M(\beta, \tau)| = q - \delta$, so that

$$|C_R(t_1)| = eq(q^2 - 1)(q - \delta), \qquad |C_R(\beta)| = |C_R(t_1, \tau)| = e(q - \delta)^2,$$
$$|C_R(\beta, \tau)| = e(q - \delta).$$

We can now apply a result of Brauer ([1], Th. 2) to conclude that

$$|R| = q^3(q^3 - \delta)(q^2 - 1)e.$$

(The integer denoted f in [1] is a root of the quadratic equation

$$|C_R(\beta)|\,|C_R(t_1, \tau)|^2 f(f + 1) = |C_R(t_1)|\,|C_R(\beta, \tau)|^2 (f - 1)^2,$$

which in this case has $f = \delta q$ as its only integral solution. Then the numbers denoted a, c, t, ϵ in [1] are easily calculated to be 1, $e(q - \delta)$, 1, δ respectively.) Since R has index 2^n in $C(v)$, and $2^n e = q - \delta$, we see that

$$|C(v)| = q^3(q^3 - \delta)(q^2 - 1)(q - \delta),$$

as asserted.

We remark that if $\delta = 1$, then Brauer's characterization of $PGL_3(q)$ [1] can be used to determine the structure of $C(v)$ completely. However, we shall not need this information.

2. The p-Structure of G

We shall determine the structure of the normalizer of a Sylow p-subgroup of G, where p is the characteristic of the field F_q.

For α in F_q, we set

$$x_i(\alpha) = \varphi_i \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \qquad (i = 1, 2, 3).$$

We shall also write

$$x_4(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\alpha \\ \alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_5(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 1 \end{pmatrix},$$

elements of J. Then, for $1 \leqslant i \leqslant 5$,

$$P_i = \{x_i(\alpha) \mid \alpha \in F_q\}$$

is a subgroup of $C(t_1)$, and the mapping

$$\alpha \to x_i(\alpha) \qquad (\alpha \in F_q) \tag{11}$$

is an isomorphism of the additive group of F_q with P_i, so that P_i is elementary Abelian of order q. P_1, P_2 and P_3 are Sylow p-subgroups of L_1, L_2 and L_3 respectively. We shall later define further subgroups P_6, P_7, P_8, P_9, each isomorphic with the additive group of F_q, and we shall denote by $P_{ijk\cdots}$ the complex $P_iP_jP_k \ldots$. Then P_{2345} is a Sylow p-subgroup of J, whose structure is determined by the fact that the mappings (11) are isomorphisms, together with the relations

$$\left.\begin{aligned} &[x_2(\alpha), x_j(\beta)] = [x_3(\alpha), x_5(\beta)] = 1 \qquad (j = 3, 4, 5),\\ &[x_3(\alpha), x_4(\beta)] = x_2(\alpha\beta^2)\, x_5(\alpha\beta),\\ &[x_4(\alpha), x_5(\beta)] = x_2(-2\alpha\beta), \end{aligned}\right\} \tag{12}$$

for all α, β in F_q. We also have

$$[x_1(\alpha), x_j(\beta)] = 1, \qquad (j = 2, 3, 4, 5), \tag{13}$$

and the subgroup P_{12345} is a Sylow p-subgroup of $C(t_1)$, of order q^5.

We shall need the elements

$$c_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (i = 1, 2, 3).$$

Since c_1 transforms $x_1(\alpha)$ into $\varphi_1\left(\begin{smallmatrix}1 & -\alpha\\ 0 & 1\end{smallmatrix}\right)$, we see that

$$P_1 \cap P_1^{c_1} = 1. \tag{14}$$

Using the symplectic basis $\{e_1, e_2, e_3, e_4\}$ of the symplectic space E on which $J = Sp_4(q)$ acts, we make the convention that $E_{ijk} \ldots$ is the subspace of E generated by $\{e_i, e_j, e_k, \ldots\}$. Then the elements of P_{2345} leave E_1 and E_{13} invariant and induce the identity transformation on E_1 and E_{13}/E_1. Since c_2c_3 transforms E_1 into E_2 and E_{13} into E_{24}, the elements of $(P_{2345})^{c_2c_3}$ leave E_2 and E_{24} invariant and induce the identity transformation on E_2 and E_{24}/E_2. Thus an element x of $P_{2345} \cap (P_{2345})^{c_2c_3}$ maps

$$e_1 \to e_1, \qquad e_2 \to e_2, \qquad e_3 \to e_3 + \beta e_1, \qquad e_4 \to e_4 + \gamma e_2,$$

for some β, γ in F_q. Since x is symplectic, $\beta = \gamma = 0$. Thus,

$$P_{2345} \cap (P_{2345})^{c_2c_3} = 1. \tag{15}$$

We compute that u normalizes P_{1235} and that c_3 normalizes P_{1245}. In fact, we have the relations

$$u : x_1(\alpha) \to x_1(\alpha), \qquad x_2(\alpha) \leftrightarrow x_3(\alpha), \qquad x_5(\alpha) \to x_5(-\alpha), \tag{16}$$

$$\begin{gathered} c_3 : x_1(\alpha) \to x_1(\alpha), \qquad x_2(\alpha) \to x_2(\alpha), \\ x_4(\alpha) \to x_5(-\alpha), \qquad x_5(\alpha) \to x_4(\alpha). \end{gathered} \tag{17}$$

The subgroup of the elements of J fixing the vectors of E_{13} is P_{235}. Hence $N_J(P_{235})$ is the set of all elements of J leaving E_{13} invariant. If $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \mu\end{smallmatrix}\right)$ is an element of $GL_2(q)$, we set

$$\varphi_4\begin{pmatrix}\alpha & \beta\\ \gamma & \mu\end{pmatrix} = \begin{pmatrix}\alpha & 0 & \beta & 0\\ 0 & \alpha' & 0 & \beta'\\ \gamma & 0 & \mu & 0\\ 0 & \gamma' & 0 & \mu'\end{pmatrix},$$

where $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \mu'\end{smallmatrix}\right)$ is the contragredient of $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \mu\end{smallmatrix}\right)$. Then this is an element of J, and φ_4 is an isomorphism of $GL_2(q)$ with a subgroup L_4 of J. Since L_4 leaves the subspace E_{13} invariant and induces the full group of nonsingular linear transformations on it, we have

$$N_J(P_{235}) = P_{235}L_4. \tag{18}$$

Since $x_4(\alpha) = \varphi_4\left(\begin{smallmatrix}1 & 0\\ \alpha & 1\end{smallmatrix}\right)$, we see that P_4 is a Sylow p-subgroup of L_4. We also remark that c_2c_3 normalizes L_4.

An easy computation shows that $C_J(P_2)$ consists of all elements of J of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ \alpha & \lambda & \beta & \gamma \\ \rho & 0 & \mu & \tau \\ \sigma & 0 & \xi & \eta \end{pmatrix}.$$

For this to be a symplectic matrix, we must have $\lambda^2 = 1$, $\lambda\rho + \mu\gamma = \beta\tau$, $\lambda\sigma + \gamma\xi = \beta\eta$, $\det \left(\begin{smallmatrix} \mu & \tau \\ \xi & \eta \end{smallmatrix}\right) = 1$. Then this element is equal to

$$\varphi_3 \begin{pmatrix} \mu & \tau \\ \xi & \eta \end{pmatrix} x_2(\alpha + \beta\gamma)\, x_5(\beta)\, x_4(-\gamma), \qquad \text{if} \qquad \lambda = 1,$$

$$t_1\varphi_3 \begin{pmatrix} -\mu & -\tau \\ -\xi & -\eta \end{pmatrix} x_2(-\alpha + \beta\gamma)\, x_5(-\beta)\, x_4(\gamma), \qquad \text{if} \qquad \lambda = -1.$$

Hence

$$C_J(P_2) = \langle t_1\rangle L_3 P_{254}\,. \tag{19}$$

Now, P_{254} may be described as the group of all elements of J leaving E_1 and E_{134} invariant and inducing the identity transformation on E_1 and E_{134}/E_1. Since $C_J(P_2)$ leaves both E_1 and E_{134} invariant, P_{254} is a normal subgroup of $C_J(P_2)$, so that the product (19) of $\langle t_1\rangle L_3$ with P_{254} is semidirect. It follows that

$$C_J(P_{23}) = \langle t_1\rangle\, C_{L_3}(P_3)\, C_{P_{254}}(P_3) = DP_{235}\,, \tag{20}$$

by the relations (12), where D is the four-group $\langle t_1\,, t_3\rangle$.

Since t_3 does not centralize P_5, we have

$$C_J(P_{235}) = P_{235}\langle t_1\rangle. \tag{21}$$

For each nonzero element λ of F_q, we set

$$h_i(\lambda) = \varphi_i \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad (i = 1, 2, 3).$$

Then, for $i = 1, 2, 3$,

$$H_i = \{h_i(\lambda) \mid 0 \neq \lambda \in F_q\}$$

is a subgroup of L_i, and the mapping

$$\lambda \to h_i(\lambda) \qquad (0 \neq \lambda \in F_q)$$

is an isomorphism of the multiplicative group of F_q with H_i, so that H_i is cyclic of order $q - 1$. The involution t_i belongs to H_i, since

$$h_i(-1) = t_i \qquad (i = 1, 2, 3).$$

Since $t_1t_2t_3 = 1$ and $[H_1, H_2] = [H_1, H_3] = [H_2, H_3] = 1$,

$$H = H_1H_2H_3$$

is Abelian of order $\frac{1}{2}(q-1)^3$. H normalizes each P_i, $1 \leqslant i \leqslant 5$. In fact we compute that the action of H on P_{12345} is given by Table I, in which the entry in the row labelled $h_i(\lambda)$ and the column labelled $x_j(\alpha)$ is the transform of $x_j(\alpha)$ by $h_i(\lambda)$.

TABLE I

	$x_1(\alpha)$	$x_2(\alpha)$	$x_3(\alpha)$	$x_4(\alpha)$	$x_5(\alpha)$
$h_1(\lambda)$	$x_1(\lambda^2\alpha)$	$x_2(\alpha)$	$x_3(\alpha)$	$x_4(\alpha)$	$x_5(\alpha)$
$h_2(\lambda)$	$x_1(\alpha)$	$x_2(\lambda^2\alpha)$	$x_3(\alpha)$	$x_4(\lambda\alpha)$	$x_5(\lambda\alpha)$
$h_3(\lambda)$	$x_1(\alpha)$	$x_2(\alpha)$	$x_3(\lambda^2\alpha)$	$x_4(\lambda^{-1}\alpha)$	$x_5(\lambda\alpha)$

We compute easily that

$$C_{L_1}(P_1) = P_1\langle t_1\rangle, \qquad N_{L_1}(P_1) = P_1H_1. \tag{22}$$

LEMMA 2.1. *$C(P_{123})$ has a normal 2-complement P_{123567} of order q^6, where $P_6 = P_5{}^s$, $P_7 = P_5^{su}$.*

Proof. By (22) and (20),

$$C(P_{123}) \cap C(t_1) = P_{1235}D, \tag{23}$$

where D is the four-group $\langle t_1, t_2\rangle$. If $t_2 = t_1{}^x$ for some x in $C(P_{123})$, then $t_1 = t_2{}^s = t_1^{xs}$, so that $xs \in C(t_1)$. Also, $P_1^{xs} = P_1{}^s = P_2$. This is impossible, since $P_1 \leqslant L_1 \lhd C(t_1)$, and $P_2 \nleqslant L_1$. Hence t_1 and t_2 are not conjugate in $C(P_{123})$. The same argument with us in place of s shows that t_1 and t_3 are not conjugate in $C(P_{123})$. Now the argument of Lemma 1.1 shows that D is a Sylow 2-subgroup of $C(P_{123})$. By Burnside's theorem, $C(P_{123})$ has a normal 2-complement K. We have

$$C_K(t_1) = P_{1235},$$

by (23). Since s and u normalize P_{123}, by (9) and (6), they normalize K. Also, $t_1{}^s = t_2$ and $t_1^{su} = t_3$, so that

$$C_K(t_2) = C_K(t_1)^s = P_{1236},$$
$$C_K(t_3) = C_K(t_1)^{su} = P_{1237},$$

where $P_6 = P_5{}^s$, $P_7 = P_5^{su}$. Since $C_{P_5}(t_2) = 1$,

$$C_K(D) = P_{123}.$$

We now apply a result of Brauer and Gorenstein and Walter ([7], p. 555), which we shall call

Brauer's Formula. If t_1, t_2, t_3 are the three involutions of a four-group D acting on a group K of odd order, then

$$|K|\,|C_K(D)|^2 = |C_K(t_1)|\,|C_K(t_2)|\,|C_K(t_3)|,$$
$$K = C_K(t_1)\,C_K(t_2)\,C_K(t_3).$$

We conclude that $|K| = q^6$ and that $K = P_{1235}P_{1236}P_{1237} = P_{123567}$. This proves the lemma.

We now set

$$x_6(\alpha) = x_5(\alpha)^s, \qquad x_7(\alpha) = x_5(\alpha)^{su}, \qquad (\alpha \in F_q), \tag{24}$$

so that the mappings $\alpha \to x_6(\alpha)$, $\alpha \to x_7(\alpha)$ are isomorphisms of the additive group of F_q with P_6 and P_7 respectively. Obviously $P_7 = P_6{}^u$. Since $su^2 = st_1 = t_2s$ and t_2 normalizes P_5, $P_6 = P_7{}^u$.

The structure of P_{123567} is determined by the following result.

LEMMA 2.2. *The subgroup P_{123567} is the normal 2-complement of $C(P_{1235})$, and is an elementary Abelian group.*

Proof. Since $C(P_{1235}) \leqslant C(P_{123})$, we deduce from Lemma 2.1 that $C(P_{1235})$ has a normal 2-complement $M = C_K(P_{1235})$, where $K = P_{123567}$. Since P_{1235} is Abelian, $M \geqslant P_{1235}$.

Suppose that $M = P_{1235}$. By (22) and (21),

$$C(P_{1235}) \cap C(t_1) = P_{1235}\langle t_1\rangle.$$

By Lemma 1.1, $\langle t_1\rangle$ is a Sylow 2-subgroup of $C(P_{1235})$. By the Frattini argument, (22) and (18),

$$N(P_{1235}) = C(P_{1235})(N(P_{1235}) \cap C(t_1))$$
$$= C(P_{1235})\,P_{1235}L_4H_1,$$

so that $P_{12354} = P_{12345}$ is a Sylow p-subgroup of $N(P_{1235})$. Now the relations (12) show that if x is any element of P_{12345} not in P_{1235}, then $C_{P_{1235}}(x) = P_1P_2$, so that any Abelian subgroup of P_{12345} not in P_{1235} has order at most q^3. Thus P_{1235} is characteristic in P_{12345}, being the only Abelian subgroup of order q^4. It follows that $N(P_{12345}) \leqslant N(P_{1235})$, so that P_{12345} is a Sylow p-subgroup of its normalizer in G. This implies that P_{12345} is a Sylow p-subgroup of G, contradicting Lemma 2.1.

Thus $M > P_{1235}$, so that M contains some nontrivial element of P_{67}, say $x_6(\alpha)\, x_7(\beta)$, where α or β is nonzero. Since H normalizes P_{1235}, H also normalizes M. From Table I, we obtain the relations

$$[x_5(\alpha), h_1(\lambda)] = 1, \qquad [x_5(\beta), h_3(\lambda)] = x_5((\lambda - 1)\beta). \tag{25}$$

Transforming by s and su, we see that

$$[x_6(\alpha), h_2(\lambda)] = 1, \qquad [x_7(\beta), h_2(\lambda)] = x_7((\lambda - 1)\beta),$$

so that M contains

$$[x_6(\alpha)\, x_7(\beta), h_2(\lambda)] = x_7((\lambda - 1)\beta),$$

for all nonzero λ in F_q. If $\beta \neq 0$, it follows that $P_7 \leqslant M$. If $\beta = 0$, then transforming the second relation (25) by s and replacing β by α, we see that M contains

$$[x_6(\alpha), h_3(\lambda)] = x_6((\lambda - 1)\alpha)$$

for all non-zero λ in F_q, so that $P_6 \leqslant M$. Since u normalizes P_{1235}, u normalizes M. Since transformation by u interchanges P_6 and P_7, M contains both P_6 and P_7 in any case, so that we have $M = P_{123567}$, and

$$P_{1235} \leqslant Z(P_{123567}).$$

Since u normalizes P_5, we see by (10) that

$$P_7{}^s = P_5^{sus} = P_5^{usu} = P_5^{su} = P_7,$$

$$[P_6, P_7] = [P_5, P_7]^s = 1.$$

Since P_6 and P_7 are Abelian, it follows that P_{123567} is Abelian. Being the direct product of $P_1, P_2, P_3, P_5, P_6, P_7$, it is elementary Abelian, This proves the lemma.

We shall use the following nonsimplicity criterion.

LEMMA 2.3. *Let X be a finite group with a Sylow 2-subgroup T isomorphic with the Sylow 2-subgroup of $Sp_4(q)$, for some odd q. If j is an involution in $Z(T)$, then*

$$X = C_X(j)\, O(X).$$

Proof. The group T is the wreath product of a generalized quaternion group with a group of order 2, i.e.

$$T = (Q_1 \times Q_2)\langle z\rangle,$$

where Q_1 and Q_2 are generalized quaternion groups of order 2^{n+1}, $z^2 = 1$, and $Q_1^z = Q_2$. Let j_1, j_2 be the involutions in Q_1, Q_2, respectively. Then $Z(T) = \langle j\rangle$, where $j = j_1, j_2$.

If x is an involution of Q_1Q_2 distinct from j, then x is j_1 or j_2, and $C_T(x) = Q_1Q_2$. The number of square roots of j in Q_1Q_2 is $(2^n + 2)^2$, while the number of square roots of each of j_1 and j_2 in Q_1Q_2 is only $2(2^n + 2)$. Hence $\langle j\rangle$ is characteristic in $C_T(x)$.

If x is an involution of T not in Q_1Q_2, then $Q_1{}^x = Q_2$, so that

$$C_T(x) = \langle x\rangle \times Q, \qquad Q = \{yy^x \mid y \in Q_1\}.$$

The group Q is of generalized quaternion type, and contains the involution $j_1j_1{}^x = j_1j_2 = j$. Since j is the only involution of $C_T(x)$ which is a square, we see that again $\langle j\rangle$ is characteristic in $C_T(x)$.

Suppose that j is conjugate in X to some involution x of T distinct from j. Since a Sylow 2-subgroup of $C_X(x) \cap C_X(j)$ is conjugate in $C_X(j)$ to a subgroup of T, we can assume, after replacing x by a suitable conjugate, that $C_T(x)$ is a Sylow 2-subgroup of $C_X(x) \cap C_X(j)$. Since $\langle j\rangle$ is characteristic in $C_T(x)$, it follows as in Lemma 1.1 that $C_T(x)$ is a Sylow 2-subgroup of $C_X(x)$, and hence that $C_T(x) = T$, since x is conjugate in X to j. This is a contradiction, since $Z(T) = \langle j\rangle$. Hence j is conjugate in X to no other involution of T. The lemma now follows from Glauberman's theorem ([*6*], Th. 1).

LEMMA 2.4. *Let $V = O(C(P_1))$. Then*

$$C(P_1) = JV, \qquad N(P_1) = JVH_1.$$

The group V/P_1 is Abelian. The order of V is q^5, and

$$V = P_{16789},$$

where $P_8 = P_4{}^s$, $P_9 = P_4^{su}$.

Proof. By (22),

$$C(P_1) \cap C(t_1) = JP_1, \qquad N(P_1) \cap C(t_1) = JP_1H_1.$$

A Sylow 2-subgroup T of J is also a Sylow 2-subgroup of $C(P_1) \cap C(t_1)$, and has $\langle t_1\rangle$ as its center. By Lemma 1.1, T is a Sylow 2-subgroup of $C(P_1)$. By Lemma 2.3, if $V = O(C(P_1))$,

$$C(P_1) = JP_1V = JV,$$

since obviously $P_1 \leqslant V$. Also, the Frattini argument and the fact that $\langle t_1\rangle$ is characteristic in T show that

$$N(P_1) = C(P_1)(N(P_1) \cap C(t_1)) = JVH_1.$$

Since $C_V(t_1) = P_1$, t_1 acts without fixed points on V/P_1. Thus t_1 inverts V/P_1, and V/P_1 is Abelian.

By (19), $C(P_2) \cap C(t_1) = L_1L_3P_{254}$. Transforming by s, we find that

$$C(P_1) \cap C(t_2) = L_2L_3P_{168},$$

where $P_8 = P_4{}^s$. Since t_2 inverts both P_5 and P_4, $t_1 = t_2{}^s$ inverts both P_6 and P_8. Since t_1 centralizes $C(P_1)$ modulo V, it follows that $P_{168} \leqslant V$. Since $L_2L_3 \leqslant J$, we see that

$$C_V(t_2) = P_{168},$$

a group of order q^3. Since $u \in C(P_1)$ and u transforms t_2 into t_3, we have

$$C_V(t_3) = P_{168}^u = P_{179},$$

where $P_9 = P_8{}^u = P_4^{su}$. By Brauer's formula, $|V| = q^5$ and

$$V = P_1P_{168}P_{179} = P_{16789},$$

since V/P_1 is Abelian, so that P_6, P_7, P_8, P_9 are permutable among themselves, modulo P_1. This proves the lemma.

We set

$$x_8(\alpha) = x_4(\alpha)^s, \qquad x_9(\alpha) = x_4(\alpha)^{su}, \qquad (\alpha \in F_q), \tag{26}$$

so that the mappings $\alpha \to x_8(\alpha)$, $\alpha \to x_9(\alpha)$ are isomorphisms of the additive group of F_q with P_8 and P_9. Transforming the relevant entries of Table I by s and su, we find the action of H on P_6, P_7, P_8, P_9. This is given in Table II.

TABLE II

	$x_6(\alpha)$	$x_7(\alpha)$	$x_8(\alpha)$	$x_9(\alpha)$
$h_1(\lambda)$	$x_6(\lambda\alpha)$	$x_7(\lambda\alpha)$	$x_8(\lambda\alpha)$	$x_9(\lambda\alpha)$
$h_2(\lambda)$	$x_6(\alpha)$	$x_7(\lambda\alpha)$	$x_8(\alpha)$	$x_9(\lambda^{-1}\alpha)$
$h_3(\lambda)$	$x_6(\lambda\alpha)$	$x_7(\alpha)$	$x_8(\lambda^{-1}\alpha)$	$x_9(\alpha)$

Also, transformation of the relations $[L_1, P_5] = [L_1, P_4] = 1$ by s and su yields the relations

$$[L_2, P_6] = [L_3, P_7] = [L_2, P_8] = [L_3, P_9] = 1. \tag{27}$$

Since $u^2 = t_1$ inverts P_6, P_7, P_8, P_9, we have

$$\begin{aligned} u : x_6(\alpha) &\to x_7(\alpha), & x_7(\alpha) &\to x_6(-\alpha), \\ x_8(\alpha) &\to x_9(\alpha), & x_9(\alpha) &\to x_8(-\alpha). \end{aligned} \tag{28}$$

Transformation of some of the relations (17) by s and su gives the relations

$$c_3 : x_6(\alpha) \to x_8(\alpha), \qquad x_8(\alpha) \to x_6(-\alpha), \tag{29}$$

$$c_2 : x_7(\alpha) \to x_9(\alpha), \qquad x_9(\alpha) \to x_7(-\alpha). \tag{30}$$

We can now determine the structure of V.

Lemma 2.5. *For α, β in F_q, the following relations hold*:

$$[x_6(\alpha), x_8(\beta)] = [x_7(\alpha), x_9(\beta)] = x_1(2\alpha\beta),$$
$$[x_i(\alpha), x_j(\beta)] = 1, \quad \textit{if } \{i,j\} \subseteq \{1, 6, 7, 8, 9\}, \{i,j\} \neq \{6, 8\} \textit{ or } \{7, 9\}.$$

Proof. Transformation of the relation $[x_5(\alpha), x_4(\beta)] = x_2(2\alpha\beta)$ given in (12) by s and su yields the first pair of relations. Since $V \leqslant C(P_1)$, obviously $[P_1, P_j] = 1$ for $j = 6, 7, 8, 9$. By the relations (27), (29), 30),

$$P_6^{c_2} = P_6\,, P_7^{c_2} = P_9\,, P_6^{c_3} = P_8\,, P_7^{c_3} = P_7\,, P_6^{c_2c_3} = P_8\,, P_7^{c_2c_3} = P_9\,. \tag{31}$$

By Lemma 2.2, $[P_6, P_7] = 1$. Transforming by c_2, c_3, c_2c_3, we find that $[P_6, P_9] = [P_7, P_8] = [P_8, P_9] = 1$. This proves the lemma.

Lemma 2.6. *Let $U = P_{123456789}$. Then U is a subgroup of G of order q^9, normalized by H, and the structure of the subgroup UH is determined by the relations*

$$[x_2(\alpha), x_8(\beta)] = 1, \qquad [x_2(\alpha), x_9(\beta)] = x_1(\alpha\beta^2)\, x_7(\alpha\beta),$$
$$[x_3(\alpha), x_8(\beta)] = x_1(\alpha\beta^2)\, x_6(\alpha\beta), \qquad [x_3(\alpha), x_9(\beta)] = 1,$$
$$[x_4(\alpha), x_6(\beta)] = x_7(-\alpha\beta), \qquad [x_4(\alpha), x_7(\beta)] = 1,$$
$$[x_4(\alpha), x_8(\beta)] = 1, \qquad [x_4(\alpha), x_9(\beta)] = x_8(\alpha\beta),$$
$$[x_5(\alpha), x_8(\beta)] = x_7(\alpha\beta), \qquad [x_5(\alpha), x_9(\beta)] = x_6(\alpha\beta),$$

the relations (12) *and* (13), *the relations of Tables* I *and* II *and Lemma* 2.5, *Lemma* 2.2, *and the known structures of the subgroups $P_i (1 \leqslant i \leqslant 9)$ and H.*

Proof. Since J has Sylow p-subgroup P_{2345} of order q^4, Lemma 2.4 shows that

$$P_{2345}V = P_{123456789} = U$$

is a Sylow p-subgroup of $C(P_1)$, of order q^9. We have seen that H normalizes each P_i, so that H normalizes U, and we have found the action of H on each P_i in Tables I and II. The structures of P_{2345} and V are given by the relations (12) and Lemma 2.5. It remains to determine the action of P_{2345} on V. We know that P_{2345} centralizes P_1, and that P_2, P_3 and P_5 centralize P_6 and P_7, by Lemma 2.2.

The first and fourth of the asserted relations follow immediately from (27).

The second and third relations are obtained by transforming by su and s the relation $[x_3(\alpha), x_4(\beta)] = x_2(\alpha\beta^2)\, x_5(\alpha\beta)$ given in (12).

For each element x of V, write $\bar{x}$ for the corresponding element of V/P_1. We shall write the Abelian group V/P_1 additively and make it into a 4-dimensional vector space over F_q by defining scalar multiplication in the obvious way:

$$\lambda(\overline{x_6(\alpha)} + \overline{x_7(\beta)} + \overline{x_8(\gamma)} + \overline{x_9(\mu)}) = \overline{x_6(\lambda\alpha)} + \overline{x_7(\lambda\beta)} + \overline{x_8(\lambda\gamma)} + \overline{x_9(\lambda\mu)},$$

where $\lambda, \alpha, \beta, \gamma, \mu \in F_q$. By Table II, $h_1(\lambda)$ acts on V/P_1 as multiplication by the scalar λ. Since J normalizes V and P_1, and centralizes H_1, J acts on V/P_1 by linear transformations. Since t_1 inverts V/P_1, this action is faithful.

If $x, y \in V$, then since V/P_1 is Abelian and $P_1 \leqslant Z(V)$, $[x, y]$ lies in P_1 and depends only on $\bar{x}$ and $\bar{y}$. Thus we have

$$[x, y] = x_1(-2f(\bar{x}, \bar{y})),$$

where f is a mapping of $(V/P_1) \times (V/P_1)$ into F_q. Lemma 2.5 shows that f is a nondegenerate alternating bilinear form, so that V/P_1 becomes a symplectic vector space over F_q, and that

$$\mathscr{B} = \{\overline{x_7(1)}, \overline{x_9(-1)}, \overline{x_6(1)}, \overline{x_8(-1)}\}$$

is a symplectic basis. We shall represent linear transformations on V/P_1 by their matrices with respect to this basis. Since J centralizes P_1, the linear transformations of V/P_1 induced by J are symplectic, and we have an isomorphism of J with the symplectic group of V/P_1. We wish to prove that this isomorphism is given by

$$x \to x, \tag{32}$$

where the x on the left is an element of J and the x on the right is the matrix of the corresponding linear transformation on V/P_1.

From Table II, (30), (27), the relations we have already proved, Lemma 2.2, and (28), we see that (32) is valid when x is $h_2(\lambda)$, c_2, $x_2(\alpha)$ or u, i.e. when x is any one of the matrices

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{33}$$

It follows that the isomorphism of J with the symplectic group of V/P_1 has the form

$$x \to \theta(x),$$

where θ is an automorphism of $Sp_4(q)$ leaving each of the matrices (33) fixed. Every automorphism of $Sp_4(q)$ is induced by a semi-similitude, i.e. a field automorphism followed by a certain type of linear transformation ([5], p. 97). If σ is the field automorphism associated with θ, then σ leaves invariant all sets $\{\lambda, \lambda^{-1}\}$, $0 \neq \lambda \in F_q$, since θ preserves the first type of matrix in (33) and transformation by a linear transformation does not change eigenvalues. Thus $\sigma^2 = 1$. If $\sigma \neq 1$, then q is a square, $q = r^2$, and σ is the automorphism $\alpha \to \alpha^r$. But if λ is a primitive root in F_q, then λ^r is not λ or λ^{-1}, so that σ does not preserve $\{\lambda, \lambda^{-1}\}$. Hence $\sigma = 1$ and θ is the automorphism induced by some linear transformation τ.

Since θ preserves the first type of matrix in (33), the matrix of τ with respect to $\mathscr{B}$ has the form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, where A and B are 2×2 matrices. Since θ preserves the second and third type of matrix in (33), A is a scalar multiple of I. Finally, since θ preserves the last of matrices (33), $A = B$. Hence τ is a scalar multiple of the identity, so that $\theta = 1$ and the isomorphism of J with the symplectic group of V/P_1 is indeed given by (32).

In particular, we see that $x_4(\alpha)$ induces the linear transformation on V/P_1 whose matrix with respect to $\mathscr{B}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\alpha \\ \alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This means that, modulo P_1, we have

$$\begin{aligned} x_4(\alpha) : x_7(\beta) &\to x_7(\beta), & x_9(\beta) &\to x_8(-\alpha\beta)\, x_9(\beta), \\ x_6(\beta) &\to x_6(\beta)\, x_7(\alpha\beta), & x_8(\beta) &\to x_8(\beta). \end{aligned} \tag{34}$$

Now, by Table II, $t_1 = h_1(-1)$ inverts P_{67}. Since t_1 centralizes P_{1235}, and P_{123567} is Abelian by Lemma 2.2,

$$[t_1, P_{123567}] = P_{67}.$$

Since L_4 normalizes P_{123567} by (18) and Lemma 2.2, and centralizes t_1, L_4 normalizes P_{67}. Since c_2c_3 normalizes L_4 and transforms P_{67} into P_{89} by (31), L_4 also normalizes P_{89}. Since $x_4(\alpha) \in L_4$, this implies that the relations (34) are valid exactly, not just modulo P_1. Thus we have the asserted relations involving $x_4(\alpha)$. Finally, transformation of two of these relations by c_3 and replacement of α by $-\alpha$ give the last two relations, by use of (17), (29), and (27). This completes the proof of the lemma.

We remark that since P_{2345} acts faithfully on V/P_1 and since we easily see from Lemma 2.5 that $Z(V) = P_1$, it follows that

$$Z(U) = P_1,$$

so that $N(U) \leqslant N(P_1)$. From Lemma 2.4 we find that

$$N(U) = N_J(P_{2345})\, VH_1 = UH.$$

Since U is a Sylow p-subgroup of UH, this implies that U is a Sylow p-subgroup of G. However, we shall not need to use this fact.

Since $|U| = q^9$, we see that every element x of U has a unique expression in the form

$$x = \prod_{i=1}^{9} x_i(\alpha_i),$$

where each α_i is an element of F_q, and the product is taken in order of increasing i. We shall call this the *standard form* of x, and refer to the ordering of the product as the *standard order*.

3. The (BN)-Pair

Using the relations (9), (16), (17), (27), (29), the definitions (24), (26), the fact that $s^2 = t_3$ inverts P_4 and P_5, and $u^2 = t_1$ inverts P_6 and P_8, and the computation

$$x_7(\alpha)^s = x_5(\alpha)^{sus} = x_5(\alpha)^{usu} = x_5(-\alpha)^{su} = x_7(-\alpha),$$

we can determine the action of s, u, c_3 on all the P_i except P_9, P_4, P_3 respectively. The result is given in Table III.

Table III

	$x_1(\alpha)$	$x_2(\alpha)$	$x_3(\alpha)$	$x_4(\alpha)$	$x_5(\alpha)$	$x_6(\alpha)$	$x_7(\alpha)$	$x_8(\alpha)$	$x_9(\alpha)$
	$x_2(\alpha)$	$x_1(\alpha)$	$x_3(\alpha)$	$x_8(\alpha)$	$x_6(\alpha)$	$x_5(-\alpha)$	$x_7(-\alpha)$	$x_4(-\alpha)$	
u	$x_1(\alpha)$	$x_3(\alpha)$	$x_2(\alpha)$		$x_5(-\alpha)$	$x_7(\alpha)$	$x_6(-\alpha)$	$x_9(\alpha)$	$x_8(-\alpha)$
c_3	$x_1(\alpha)$	$x_2(\alpha)$		$x_5(-\alpha)$	$x_4(\alpha)$	$x_8(\alpha)$	$x_7(\alpha)$	$x_6(-\alpha)$	$x_9(\alpha)$

We now set

$$N = \langle H, s, u, c_3 \rangle.$$

This subgroup normalizes H and contains $c_2 = c_3{}^u$ and $c_1 = c_2{}^s$. Also, we set

$$B = UH.$$

By Lemma 2.6, the structure of B is uniquely determined. We shall show that the structure of N is also uniquely determined.

LEMMA 3.1. *The structure of* $N = \langle H, s, u, c_3 \rangle = \langle H, s, u, c_1, c_2, c_3 \rangle$ *is determined by the structure of* H *together with the relations*

$$\left.\begin{array}{lll} s : h_1(\lambda) \leftrightarrow h_2(\lambda), & h_3(\lambda) \to h_3(\lambda), & \\ u : h_1(\lambda) \to h_1(\lambda), & h_2(\lambda) \leftrightarrow h_3(\lambda), & \\ c_3 : h_1(\lambda) \to h_1(\lambda), & h_2(\lambda) \to h_2(\lambda), & h_3(\lambda) \to h_3(\lambda^{-1}), \end{array}\right\} \tag{35}$$

$$\left.\begin{array}{c} c_i^2 = t_i \quad (i = 1, 2, 3), \qquad [c_1, c_2] = [c_1, c_3] = [c_2, c_3] = 1, \\ s^2 = t_3, \quad u^2 = t_1, \qquad (su)^3 = 1, \\ s : c_1 \leftrightarrow c_2, \quad c_3 \to c_3, \qquad u : c_1 \to c_1, \quad c_2 \leftrightarrow c_3. \end{array}\right\} \tag{36}$$

The quotient group N/H *has order* 48 *and is isomorphic with the wreath product of a group of order* 2 *with the symmetric group of degree* 3. *Also,*

$$B \cap N = H.$$

Proof. The relations (35), (36) hold, by the structure of $C(t_1)$ and the properties of s (Lemma 1.3 and (9)).

Taken modulo H, the relations (36) clearly become defining relations for the wreath product of a group of order 2 with the symmetric group of degree 3. Since this wreath product has order 48, $|N/H| \leq 48$. From the structure of $C(t_1)$, $|\langle H, c_1, c_2, c_3\rangle/H| = 8$, and $\langle H, c_1, c_2, c_3\rangle$ centralizes $D = \langle t_1, t_2\rangle$. Since s and u induce automorphisms of D which generate the full automorphism group of order 6, we see that $|N/H| \geq 48$. Hence $|N/H| = 48$, and we have a complete set of defining relations for N. Also,

$$C_N(D) = \langle H, c_1, c_2, c_3\rangle,$$

so that

$$|C_N(D)| = 8|H| = 4(q-1)^3,$$

which is prime to p.

Since H normalizes U and N normalizes H,

$$[U \cap N, H] \leqslant [U, H] \cap [N, H] \leqslant U \cap H = 1,$$

so that $U \cap N \leqslant C_N(H) \leqslant C_N(D)$. Since $U \cap N$ is a p-group, it follows that $U \cap N = 1$. Since $B = UH$ and $B \cap N \geqslant H$, we must have $B \cap N = H$. This proves the lemma.

We remark that the relations (36) can be replaced by the shorter set

$$s^2 = t_3, \qquad u^2 = t_1, \ c_3^2 = t_3, \qquad (su)^3 = (sc_3)^2 = 1, \qquad (uc_3)^4 = t_1,$$

which show that N/H is the group denoted [3, 4] in ([3], p. 37). An alternative description of N/H is that it is isomorphic with the direct product of a group of order 2 with the symmetric group of degree 4.

We set

$$W = N/H,$$

$$r_1 = sH, \qquad r_2 = uH, \qquad r_3 = c_3H.$$

Then r_1, r_2, r_3 are involutions generating W. For each element w of W we fix a coset representative $\omega(w)$ in N, so that

$$w = \omega(w)H.$$

If K is any subgroup containing H, then $\omega(w)K$ and $K\omega(w)$ are independent of the choice of the coset representative, and may be written wK and Kw. Similarly, if K is any subgroup normalized by H, $K^{\omega(w)}$ may be written K^w.

A definite choice of the $\omega(w)$ can be made as follows. Choose

$$\omega(1) = 1, \qquad \omega(r_1) = s, \qquad \omega(r_2) = u, \qquad \omega(r_3) = c_3.$$

For any other element w of W, take all the representations of w in the form

$$w = r_{i_1} r_{i_2} \cdots r_{i_k}$$

for which the length k is minimal, and choose that one for which $(i_1, i_2, \ldots, i_k)$ is first in lexicographical order. Then set

$$\omega(w) = \omega(r_{i_1})\, \omega(r_{i_2}) \cdots \omega(r_{i_k}).$$

The group W has center of order 2, generated by the coset

$$w_0 = (r_1 r_2 r_3)^3$$

of $c_1c_2c_3$.

LEMMA 3.2. $U \cap U^{w_0} = 1.$

Proof. Set

$$K = U \cap U^{w_0} = U \cap U^{c_1c_2c_3}.$$

Since $D = \langle t_1, t_2\rangle$ normalizes U and $c_1c_2c_3$ centralizes D, D normalizes K and we have

$$\begin{aligned} C_K(t_1) &= C_U(t_1) \cap C_U(t_1)^{c_1c_2c_3} \\ &= P_{12345} \cap (P_{12345})^{c_1c_2c_3}. \end{aligned}$$

Applying (14) and (15), we see that

$$C_K(t_1) = 1.$$

By Table III, $(P_{12345})^s$ and $(P_{12345})^{su}$ are contained in U. Since $(P_{12345})^s \leqslant C(t_1)^s = C(t_2)$ and $(P_{12345})^{su} \leqslant C(t_1)^{su} = C(t_3)$, and the Sylow p-subgroups of $C(t_2)$ and $C(t_3)$ have order q^5, we see that

$$C_U(t_2) = C_U(t_1)^s, \qquad C_U(t_3) = C_U(t_1)^{su}.$$

Since s and su centralize $c_1c_2c_3$,

$$C_K(t_2) = C_U(t_2) \cap C_U(t_2)^{c_1c_2c_3} = (C_U(t_1) \cap C_U(t_1)^{c_1c_2c_3})^s = 1,$$

$$C_K(t_3) = C_U(t_3) \cap C_U(t_3)^{c_1c_2c_3} = (C_U(t_1) \cap C_U(t_1)^{c_1c_2c_3})^{su} = 1.$$

Thus, t_1, t_2 and $t_3 = t_1t_2$ all invert K. Since K has odd order, this implies that $K = 1$. This proves the lemma.

We set

$$P_{-i} = P_i^{w_0}, \qquad 1 \leqslant i \leqslant 9,$$

$$V = U^{w_0} = \prod_{i=1}^{9} P_{-i}.$$

Since H normalizes P_i and w_0 normalizes H, H normalizes each P_{-i} and hence normalizes V. Also, since w_0 is an involution, we have

$$P_{-i}^{w_0} = P_i \quad (1 \leqslant i \leqslant 9), \qquad V^{w_0} = U.$$

LEMMA 3.3. *For each $i = \pm 1, \pm 2, \ldots, \pm 9$, and each element w of W*

$$P_i^{w} = P_j,$$

for some $j = \pm 1, \pm 2, \ldots, \pm 9$. In particular,

$$P_9^{r_1} = P_{-9}, \qquad P_4^{r_2} = P_{-4}, \qquad P_3^{r_3} = P_{-3}.$$

Proof. It is enough to prove the first assertion when w is r_1, r_2 or r_3. Since w_0 lies in the center of W and transforms P_i into P_{-i} for all i, we may assume that $i > 0$. By Table III, we need only consider $P_9^{r_1}$, $P_4^{r_2}$ and $P_3^{r_3}$.

Since $w_0 \in Z(W)$, we have

$$w_0 = (r_3r_2r_1)^3 = (r_3r_1r_2)^3 = (r_1r_2r_3)^3.$$

Using these three expressions for w_0 in turn, and applying Table III, we easily compute that

$$P_{-9} = P_9^{w_0} = P_9^{r_1}, \qquad P_{-4} = P_4^{w_0} = P_4^{r_2}, \qquad P_{-3} = P_3^{w_0} = P_3^{r_3},$$

so that the lemma is proved.

If $P_i^{w} = P_j$, we write $i = w(j)$. For each w in W we have uniquely determined sets of integers

$$E'_w = \{i \mid 1 \leqslant i \leqslant 9, w(i) > 0\},$$
$$E_w = \{i \mid 1 \leqslant i \leqslant 9, w(i) < 0\}.$$

Also, we set

$$U'_w = U \cap U^w, \qquad U_w = U \cap V^w.$$

In particular, writing $U_i = U_{r_i}$ $(i = 1, 2, 3)$, we have

$$U_1 = P_9, \qquad U_2 = P_4, \qquad U_3 = P_3, \tag{37}$$

by Table III and Lemma 3.3.

LEMMA 3.4. *If $w \in W$, then*

$$U'_w = \Pi\{P_i \mid i \in E'_w\}, \qquad U_w = \Pi\{P_i \mid i \in E_w\},$$

where the products are taken in standard order, and

$$U = U'_w U_w, \qquad U'_w \cap U_w = 1.$$

Proof. By Lemma 3.2, $U'_w \cap U_w = 1$, so that $| U'_w U_w | = | U'_w | \, | U_w |$. Clearly

$$U'_w \supseteq \Pi\{P_i \mid i \in E'_w\},$$
$$U_w \supseteq \Pi\{P_i \mid i \in E_w\}.$$

If $| E'_w | = n'(w)$ and $| E_w | = n(w)$, then the sets on the right contain $q^{n'(w)}$ and $q^{n(w)}$ elements respectively, by the uniqueness of the standard form of an element of U. If either inclusion were proper, $| U'_w U_w |$ would exceed $q^{n'(w)+n(w)} = q^9$, which is impossible, since $U'_w U_w \subseteq U$. Hence the inclusions are equalities, and $| U'_w U_w | = q^9$, so that $U'_w U_w = U$.

LEMMA 3.5. *If $i = 1, 2, 3$, then*

$$U_i^{r_i} \subseteq B \cup Br_i U_i .$$

Proof. First consider the case $i = 2$. Then, $U_2^{r_2} = P_4{}^u$ is contained in the subgroup L_4 of J, which is isomorphic with $GL_2(q)$. This group has the Bruhat decomposition

$$L_4 = P_4 H_2 H_3 \cup P_4 H_2 H_3 u P_4$$

(cf. [2], p. 34). Since $P_4H_2H_3 \subseteq UH = B$, we have the desired result for $i = 2$.

Since $u^{su} = (su)^{-1}\, usu = (su)^{-1}\, sus = s = \omega(r_1)$, by (10), and $P_4^{su} = P_9 = U_1$, we see that

$$U_1^{r_1} \subseteq L_4^{su} = P_9H_1H_2 \cup P_9H_1H_2sP_9\,,$$

so that we have the result for $i = 1$.

Finally, for $i = 3$, by the Bruhat decomposition for $L_3 \approx SL_2(q)$,

$$U_3^{r_3} = P_3^{c_3} \subseteq L_3 = P_3H_3 \cup P_3H_3c_3P_3\,,$$

so that we have the result for $i = 3$.

Lemma 3.6. *If $w \in W$, $i = 1, 2, 3$, then*

$$r_iBw \subseteq BwB \cup Br_iwB.$$

Proof. By Lemma 3.4 and the definition of U'_{r_i}, we have

$$r_iBw = r_iHUw = Hr_iU'_{r_i}U_iw \subseteq HUr_iU_iw = Br_iU_iw.$$

By (37), $U_i = P_j$, where $j = 9, 4, 3$ according as $i = 1, 2, 3$. If $U_i^w \leqslant U$, then

$$Br_iU_iw = Br_iwU_i^w \subseteq Br_iwU \subseteq Br_iwB.$$

Thus we may assume that $U_i^w \nleqslant U$.

By Lemma 3.3, $U_i^w \leqslant V$, and $U_i^{r_i} = P_j^{r_i} = P_{-j} = U_i^{w_0}$. Hence

$$U_i^{r_iw} = U_i^{w_0w} = U_i^{ww_0} \leqslant V^{w_0} = U,$$

since $w_0 \in Z(W)$. Applying Lemma 3.5, we have

$$Br_iU_iw = BU_i^{r_i}r_iw \subseteq Br_iw \cup Br_iU_ir_iw.$$

Since $Br_iw \subseteq Br_iwB$ and

$$Br_iU_ir_iw = BwU_i^{r_iw} \subseteq BwU \subseteq BwB,$$

we have the asserted result.

Lemma 3.7. *Let $G_0 = BNB$. Then G_0 is a subgroup of G, and is the disjoint union of the* 48 *double cosets BwB, $w \in W$.*

Proof. This follows from Lemma 3.6 by a theorem of Tits ([*11*], Th. 1).

LEMMA 3.8. *Every element of G_0 has a unique expression in the form $b\omega(w)x$, where $b \in B$, $w \in W$, $x \in U_w$. The multiplication table of G_0 is uniquely determined.*

Proof. The existence and uniqueness of the "normal form" is proved in the usual way ([*2*], p. 42), by using Lemmas 3.2, 3.4. Because of the unique determination of the structures of B and N in Lemmas 2.6, 3.1, and the uniquely determined Table III, the normal form of the product of two elements of G_0 given in normal form is uniquely determined, so that the multiplication table of G_0 is uniquely determined (cf. [*8*], §8).

LEMMA 3.9. $G_0 = G$.

Proof. By the Bruhat decomposition of $SL_2(q)$ previously mentioned,

$$L_1 = \langle P_1, H_1, c_1 \rangle \leqslant G_0 .$$

By a result of Dickson on generators of $Sp_{2m}(q)$ ([*4*], p. 92),

$$J = \langle c_2, c_3, P_2, P_3, P_5 \rangle \leqslant G_0 .$$

Hence $C(t_1) \leqslant G_0$. Since $[t_1, s] = t_3$ has even order, $G_0 \neq C(t_1)\, O(G_0)$. Thus G_0 satisfies the hypotheses (1), (2). By Lemma 1.2, G_0 must have two classes of involutions, which must be

$$K_1' = K_1 \cap G_0 , \qquad K_2' = K_2 \cap G_0 .$$

Applying Lemma 1.6, we see that if x is any involution of G_0 then $C_{G_0}(x) = C(x)$. Since G has two classes of involutions, G_0 must contain all the involutions of G ([*9*], Lemma 1). In particular, $K_1' = K_1$, so that

$$| G_0 | = | K_1' | \, | C_{G_0}(t_1)| = | K_1 | \, | C(t_1)| = | G |.$$

This proves the lemma.

LEMMA 3.10. *G is isomorphic with $PSp_6(q)$.*

Proof. By Lemmas 3.8, 3.9, the conditions (1) and (2) uniquely determine the multiplication table of G. Since $PSp_6(q)$ satisfies (1) and (2), we have

$$G \approx PSp_6(q).$$

This completes the proof of the theorem.

References

1. Brauer, R. Investigation on groups of even order, II. *Proc. Nat. Acad. Sci. U.S.A.* **55** (1966), 254–259.

2. Chevalley, C. Sur certains groupes simples. *Tôhoku Math. J.* (2) **7** (1955), 14–66.

3. Coxeter, H. S. M. and Moser, W. O. "Generators and Relations for Discrete Groups," 2nd ed. Springer, Berlin, 1965.

4. Dickson, L. E. "Linear Groups." Dover, New York, 1958.

5. Dieudonné, J. "La Géométrie des Groupes Classiques," 2nd ed. Springer, Berlin, 1963.

6. Glauberman, G. Central elements in core-free groups. *J. Algebra* **4** (1966), 403–420.

7. Gorenstein, D. and Walter, J. H. On finite groups with dihedral Sylow 2-subgroups. *Ill. J. Math.* **6** (1962), 553–593.

8. Suzuki, M. On characterizations of linear groups V. To appear.

9. Suzuki, M. Two characteristic properties of (ZT)-groups. *Osaka Math. J.* **15** (1963), 143–150.

10. Thompson, J. G. Nonsolvable finite groups all of whose local subgroups are solvable. *Bull. Am. Math. Soc.* **74** (1968), 383–437.

11. Tits, J. Théorème de Bruhat et sous-groupes paraboliques. *C. R. Acad. Sci. Paris* **254** (1962), 2910–2912.

12. Wong, W. J. A characterization of the finite projective symplectic groups $PSp_4(q)$. *Trans. Am. Math. Soc.* To appear.